Yau 2025 Applied Math – Individual Overall Contest

1. Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a scalar $\alpha > 0$, the proximal operator $\mathbf{prox}_{\alpha,f}$ of f is defined as the mapping from a point $x \in \mathbb{R}^n$ to the unique solution of the minimization problem:

$$\min_{y\in\mathbb{R}^n} \; f(y) + \frac{1}{2\alpha}\|y - x\|_2^2.$$

   (For the solution to always exist, we technically assume f is closed and proper, but these conditions are not crucial for this problem.)

   Many functions, such as $\|x\|_1$ and $\|x\|_2$, have explicit forms for their proximal operators. For a fixed $\alpha > 0$ and $n \in \mathbb{N}$, perform the following:

   (a) Derive the explicit formula for $\mathbf{prox}_{\alpha,f}$ when $f(x) = \|x\|_1$.

   (b) Derive the explicit formula for $\mathbf{prox}_{\alpha,g}$ when $g(x) = \|x\|_2$.

   (c) Prove that $\mathbf{prox}_{f+g} = \mathbf{prox}_g \circ \mathbf{prox}_f$ holds for $f(x) = \|x\|_1$ and $g(x) = \|x\|_2$.

2. Consider the boundary value problem:

$$\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0, \quad x \in (0,1),\ \varepsilon > 0 \\ y(0) = 0,\ y(1) = 1. \end{cases}$$

   Derive the leading-order uniform approximation of the solution as $\varepsilon \to 0$.